Pôle Universitaire Léonard de Vinci
ESI CS 3906 – Contrôle actif de structures

S10 – Année 2006-2007
Promotion ESILV 2007

Interrogation du 7/12/2006

Corrigé

Exercice I

1. On a

$$\int_\Omega \operatorname{div}((D\nabla w)\nabla\hat{w} - \Delta w \nabla\hat{w})\, d\Omega = \int_\Gamma (\langle (D\nabla w)\nabla\hat{w}, n\rangle - \Delta w\, \langle \nabla\hat{w}, n\rangle)\, d\Gamma$$

2. On a

$$\nabla w = \begin{pmatrix} \frac{\partial^2 w}{\partial x_1^2} \\ \frac{\partial^2 w}{\partial x_1 \partial x_2} \end{pmatrix}$$

ainsi

$$D\nabla w = \begin{pmatrix} \frac{\partial^2 w}{\partial x_1^2} & \frac{\partial w}{\partial x_1} \\ \frac{\partial^2 w}{\partial x_1 \partial x_2} & \frac{\partial^2 w}{\partial x_2^2} \end{pmatrix}$$

donc

$$(D\nabla w)t = \begin{pmatrix} -n_2 \frac{\partial^2 w}{\partial x_1^2} + n_1 \frac{\partial^2 w}{\partial x_1 \partial x_2} \\ -n_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} + n_1 \frac{\partial^2 w}{\partial x_2^2} \end{pmatrix}$$

d'où

$$\langle (D\nabla w)t, n\rangle = -n_1 n_2 \frac{\partial^2 w}{\partial x_1^2} + n_1^2 \frac{\partial^2 w}{\partial x_1 \partial x_2} - n_2^2 \frac{\partial^2 w}{\partial x_1 \partial x_2} + n_1 n_2 \frac{\partial^2 w}{\partial x_2^2}$$

il en résulte

$$\langle (D\nabla w)t, n\rangle = (n_1^2 - n_2^2) \frac{\partial^2 w}{\partial x_1 \partial x_2} + n_1 n_2 \left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2} \right)$$

3. On a

$$(D\nabla w)n = \begin{pmatrix} n_1 \frac{\partial^2 w}{\partial x_1^2} + n_2 \frac{\partial w}{\partial x_1} \\ n_1 \frac{\partial^2 w}{\partial x_1 \partial x_2} + n_2 \frac{\partial^2 w}{\partial x_2^2} \end{pmatrix}$$

ainsi

$$\langle (D\nabla w)n, n\rangle = n_1^2 \frac{\partial^2 w}{\partial x_1^2} + 2 n_1 n_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} + n_2^2 \frac{\partial^2 w}{\partial x_2^2}$$

donc

$$\langle (D\nabla w)n, n\rangle - \Delta w = (n_1^2 - 1) \frac{\partial^2 w}{\partial x_1^2} + 2 n_1 n_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} + (n_2^2 - 1) \frac{\partial^2 w}{\partial x_2^2}$$

or n est unitaire donc $\|n\|^2 = 1$ donc $n_1^2 + n_2^2 = 1$, ainsi $n_1^2 - 1 = -n_2^2$ et $n_2^2 - 1 = -n_1^2$, par suite

$$\langle (D\nabla w)n, n\rangle - \Delta w = -n_2^2 \frac{\partial^2 w}{\partial x_1^2} + 2 n_1 n_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} - n_1^2 \frac{\partial^2 w}{\partial x_2^2}$$

4. On a $\nabla \hat{w} = \langle \nabla \hat{w}, n\rangle\, n + \langle \nabla \hat{w}, t\rangle\, t$ ainsi

$$\langle (D\nabla w)\nabla \hat{w}, n\rangle - \Delta w \,\langle \nabla \hat{w}, n\rangle = \langle \nabla \hat{w}, n\rangle\, \langle (D\nabla w)n, n\rangle\, \langle \nabla \hat{w}, t\rangle\, \langle (D\nabla w)t, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle$$

ainsi

$$\begin{aligned}\int_\Gamma (\langle (D\nabla w)\nabla \hat{w}, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle)\, d\Gamma \\ = \int_\Gamma (\langle \nabla \hat{w}, n\rangle\, \langle (D\nabla w)n, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle)\, d\Gamma + \int_\Gamma (\langle \nabla \hat{w}, t\rangle\, \langle (D\nabla w)t, n\rangle)\, d\Gamma\end{aligned}$$

en vertu des questions précédentes, nous obtenons

$$\begin{aligned}\int_\Gamma (\langle (D\nabla w)\nabla \hat{w}, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle)\, d\Gamma \\ = \int_\Gamma \langle \nabla \hat{w}, n\rangle \left(-n_2^2 \frac{\partial^2 w}{\partial x_1^2} + 2n_1 n_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} - n_1^2 \frac{\partial^2 w}{\partial x_2^2}\right) d\Gamma \\ + \int_\Gamma \langle \nabla \hat{w}, t\rangle \left((n_1^2 - n_2^2)\frac{\partial^2 w}{\partial x_1 \partial x_2} + n_1 n_2 \left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}\right)\right) d\Gamma\end{aligned}$$

ainsi

$$\begin{aligned}\int_\Gamma (\langle (D\nabla w)\nabla \hat{w}, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle)\, d\Gamma \\ = \int_\Gamma \frac{\partial \hat{w}}{\partial n} B_1 w \, d\Gamma + \int_\Gamma \frac{\partial \hat{w}}{\partial t} \left((n_1^2 - n_2^2)\frac{\partial^2 w}{\partial x_1 \partial x_2} + n_1 n_2 \left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}\right)\right) d\Gamma\end{aligned}$$

ainsi

$$\begin{aligned}\int_\Gamma (\langle (D\nabla w)\nabla \hat{w}, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle)\, d\Gamma \\ = \int_\Gamma \frac{\partial \hat{w}}{\partial n} B_1 w \, d\Gamma - \int_\Gamma \hat{w} \frac{\partial}{\partial t} \left((n_1^2 - n_2^2)\frac{\partial^2 w}{\partial x_1 \partial x_2} + n_1 n_2 \left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}\right)\right) d\Gamma\end{aligned}$$

il en résulte

$$\int_\Gamma (\langle (D\nabla w)\nabla \hat{w}, n\rangle - \Delta w\, \langle \nabla \hat{w}, n\rangle)\, d\Gamma = \int_\Gamma \left(\frac{\partial \hat{w}}{\partial n} B_1 w - \hat{w} B_2 w\right) d\Gamma$$

5. On a

$$2\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 \hat{w}}{\partial x_1 \partial x_2} - \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 \hat{w}}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 \hat{w}}{\partial x_1^2} = \operatorname{div}((D\nabla w)\nabla \hat{w} - \Delta w \nabla \hat{w})\, d\Omega$$

la question 1 et la question 4 conduisent à

$$\int_\Omega \left(2\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 \hat{w}}{\partial x_1 \partial x_2} - \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 \hat{w}}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 \hat{w}}{\partial x_1^2}\right) d\Omega = \int_\Gamma \left(\frac{\partial \hat{w}}{\partial n} B_1 w - \hat{w} B_2 w\right) d\Gamma$$

Exercice II

1. On a

$$U(x_1, x_2, x_3) = \begin{pmatrix} (\exp(-t) - 1)x_1 \\ 0 \\ 0 \end{pmatrix}$$

2. On a

$$E_c(t) = \frac{\rho}{2} \int_P \|U'(t)\|^2 \, d\Omega$$

ce qui conduit à

$$E_c(t) = \frac{\rho}{2} \int_P \exp(-2t) x_1^2 d\Omega$$

c'est-à-dire

$$E_c(t) = \frac{\rho}{2} \exp(-2t) \int_0^L \int_0^l \int_{-h/2}^{h/2} x_1^2 d\Omega$$

ainsi

$$E_c(t) = \frac{\rho L^3 l h}{6} \exp(-2t)$$

3. On a

$$DU = \begin{pmatrix} \exp(-t) - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

le tenseur ϵ linéarisé est donc

$$\epsilon = \frac{1}{2}(\, {}^t DU + DU) = \begin{pmatrix} \exp(-t) - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ainsi

$$\sigma = \begin{pmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}[\exp(-t) - 1] & 0 & 0 \\ 0 & \frac{E\nu}{(1+\nu)(1-2\nu)}[\exp(-t) - 1] & 0 \\ 0 & 0 & \frac{E\nu}{(1+\nu)(1-2\nu)}[\exp(-t) - 1] \end{pmatrix}$$

donc

$$\sigma \cdot \cdot \epsilon = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}[\exp(-t) - 1]^2$$

ainsi

$$E_p(t) = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}[\exp(-t) - 1]^2 L l h$$